Inverse characterization of sound absorbing media using one dimensional analytical Biot's poroelasticity theory solutions

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ABSTRACT

he design of vibro–acoustic systems incorporating poroelastic materials, primarily aimed at noise reduction or cancellation, requires knowledge of several material parameters. These material parameters usually are possible to be defined using some regular test in a standard acoustic laboratory. Here, we study an inverse technique for the characterization of poroelastic materials based on Biot´s theory of poroelasticity. Experimental setups for such a procedure typically consist of a configuration of several sequentially positioned layers, each of which could be a poroelastic medium, a fluid, or an elastic solid. As an indicative example we mention a configuration formed by a fluid, two porous layers saturated by a fluid, a solid layer and a second fluid. The configurations considered here result in a one-dimensional problem in the longitudinal direction, free of external forces per unit volume, stationary, and in the frequency domain ω. The problem represents the ideal conditions of the Kundt's or impedance tube. In our work we take advantage of analytical solutions for the one-dimensional case of the poroelasticity´s boundary value problem. Based on previous works of one of the authors (LSR), the macroscopic model is valid from low to high frequencies; therefore, the analytical solutions are valid from low to high frequencies. Utilizing these solutions we develop a java-based toolkit that solves genelar case of an indefinite finite number of layers multidomain problem and calculates acoustical indicators, e.g. the surface impedance, the reflection coefficient or the absorption coefficient, important for poroelastic material characterization in vibro-acoustic applications. Following that we set a minimization problem in order to approximate the material parameters using the measured acoustical indicators. For the minimization procedure we use an evolutionary algorithm, namely the differential evolution, which is a gradient free algorithm appropriate for global optimization. Finally, examples that validate the current approach are presented.

Αντίστροφος χαρακτηρισμός ηχοαπορροφητικών μέσων με χρήση μονοδιάστατων αναλυτικών λύσεων της θεωρίας πορoελαστικότητας του Biot

ΠΕΡΙΛΗΨΗ

Ο σχεδιασμός δονητικοακουστικών συστημάτων που αποτελούνται από πορoελαστικό υλικό, με κύριο στόχο τη μείωση του θορύβου, απαιτεί τη γνώση πολλών παραμέτρων του υλικού. Οι παράμετροι αυτές είναι συνήθως εφικτό να καθοριστούν με τη χρήση κάποιας συνήθους δοκιμής σε τυπικό εργαστήριο ακουστικής. Εδώ, μελετάμε μια αντίστροφη τεχνική για τον χαρακτηρισμό των πορoελαστικών υλικών που βασίζεται στη θεωρία του Biot. Οι πειραματικές διατάξεις για μια τέτοια διαδικασία αποτελούνται συνήθως από κάποια πολυστρωματική διαμόρφωση, όπου καθεμία στρώση στη γενική περίπτωση δύναται να είναι ένα πορoελαστικό μέσο, κάποιο ρευστό ή ένα ελαστικό στερεό. Ως ενδεικτικό παράδειγμα αναφέρεται η περίπτωση που σχηματίζεται από ένα ρευστό, δύο πορώδη στρώματα κορεσμένα από κάποιο ρευστό, ένα στερεό στρώμα και ένα δεύτερο ρευστό. Οι διατάξεις που εξετάζονται εδώ οδηγούν σε ένα μονοδιάστατο πρόβλημα στη διαμήκη διεύθυνση, απουσία εξωτερικών δυνάμεων ανά μονάδα όγκου, ενώ η μαθηματική περιγραφή τίθεται στο πεδίο των συχνοτήτων ω. Το πρόβλημα αντιπροσωπεύει τις ιδανικές συνθήκες του σωλήνα Kundt ή σωλήνα εμπέδησης. Στην εργασία αυτή χρησιμοποιούμε αναλυτικές λύσεις για τη μονοδιάστατη περίπτωση του προβλήματος συνοριακών τιμών της πορoελαστικότητας. Με βάση προηγούμενες μελέτες ενός εκ των συγγραφέων (LSR), το μακροσκοπικό μοντέλο και οι αντίστοιχες αναλυτικές λύσεις, ισχύουν για μεγάλο συχνοτικό εύρος από χαμηλές έως υψηλές συχνότητες. Αξιοποιώντας αυτές τις λύσεις, αναπτύσσουμε μια εργαλειοθήκη σε ένα περιβάλλον εργασίας σε java που αντιμετωπίζει τη γενική περίπτωση του προβλήματος πολλαπλών πεδίων για ένα απροσδιόριστο πεπερασμένο αριθμό στρωμάτων και υπολογίζει κατάλληλους ακουστικούς δείκτες, π.χ. την επιφανειακή αντίσταση, τον συντελεστή ανάκλασης ή τον συντελεστή απορρόφησης, σημαντικούς για τον χαρακτηρισμό των πορoελαστικών υλικών σε δονητικοακουστικές εφαρμογές. Στη συνέχεια θέτουμε ένα πρόβλημα ελαχιστοποίησης προκειμένου να προσεγγίσουμε τις παραμέτρους του υλικού χρησιμοποιώντας δεδομένους ακουστικούς δείκτες. Για τη διαδικασία ελαχιστοποίησης χρησιμοποιούμε έναν εξελικτικό αλγόριθμο, (differential evolution), ο οποίος δεν χρησιμοποιεί τις κλίσεις της συνάρτησης ελαχιστοποίησης και είναι κατάλληλος για προβλήματα εύρεσης καθολικού ελαχίστου. Τέλος, παρουσιάζονται παραδείγματα που τεκμηριώνουν την προτεινόμενη μεθοδολογία.

1 Introduction

It is a common procedure to treat acoustic problems in poroelastic media saturated by air by neglecting the two waves that propagate mainly through the solid skeleton. Therefore, models that account only for the wave propagating through the fluid interacting with the solid skeleton are typically adopted.. One of the most common such models is that of Biot, Johnson and Allard [1], which accounts for the air compressibility and the temperature effects. Nevertheless, such an approximation would not be acceptable if the saturated fluid is a liquid. That is even more pronounced when inverse characterization of materials is sought, because in such cases we need the full volume of information that is embedded into the response of such mediums. A typical example are porous metals immersed in noisy liquid environments.

In the present work we trace back to Biot's more general theory of propagation of elastic waves in a fluid-saturated porous solid, since the waves that propagate through the solid skeleton can not be considered to be negligible when the saturated fluid is a liquid. We follow previous publications of the second author (LSR) on analytical solutions for the general Biot's theory in the range from low to high frequency content when the medium can be considered to be one dimensional. Having faced situations where the sequence of successive material parts lead to extremely intricate and lengthy analytical solutions we proceed by developing a numerical framework that deploys analytical solution for single one dimensional domains arriving at a toolkit capable to confront effectively a indefinite number of sequential successive materials that constitute a coherent wave propagation medium. We then use the developed numerical framework to perform the inverse characterization of the materials through a virtual experiment.

2 Implementation

Kinematics equations in Biot's theory express the relation between the macroscopic displacement of the solid u_i and that of the fluid v_i, with the macroscopic deformations of the solid ϵ_{ij} and that of the fluid e_{ij}, respectively,

$$\begin{aligned}\epsilon_{ij} &= \frac{1}{2}(u_{i,j} + u_{j,i}),\\ e_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}).\end{aligned} \tag{1}$$

Furthermore, Biot's constitutive model of poroelasticity establishes the behaviour of a fluid-filled porous solid,

$$\begin{aligned}\sigma_{ij} &= \delta_{ij}(\lambda'\epsilon_{kk} + Be) + 2\mu\epsilon_{ij},\\ s &= B\epsilon_{kk} + Ae,\end{aligned} \tag{2}$$

where, σ_{ij} the macroscopic stress of the solid skeleton, while s is the negative value of the hydrostatic pressure of the fluid and e is the cubic dilatation of the fluid, both at the macroscopic level. Then, λ' and μ are constants connected to Lame's parameters, that relate the stress and strain of the solid skeleton. The elastic constant, A, represents the modulus of elasticity that relates the spherical waves components of the macroscopic strain and stress of the fluid. Finally, B serves for the static coupling of fluid and the solid skeleton. Equations of motion for external forces per unit volume over the solid skeleton f_i and the fluid ϕ_i can be given as,

$$\begin{aligned} \sigma_{ij} + f_i &= b_{ij}(\dot{u}_j - \dot{v}_j) + \rho_{11}\ddot{u}_i + \rho_{12}\ddot{v}_i, \\ s_j + \phi_i &= b_{ij}(\dot{v}_j - \dot{u}_j) + \rho_{12}\ddot{u}_i + \rho_{22}\ddot{v}_i. \end{aligned} \tag{3}$$

where, ρ_{11}, ρ_{12} and ρ_{22} are directly related with the open porosity β, the density of the base material of the solid skeleton ρ_s, the density of the fluid ρ_f and the apparent density of the dynamics coupling ρ_α. For isotropic permeability of the solid skeleton, the flow resistivity b_{ij} reduces to a single constant $\delta_{ij}b$. After replacement of the stresses by the strains in the equilibrium equations of eq. (3) and using the constitutive relations of eq. (2), we arrive at,

$$\begin{aligned} (\lambda' + \mu)u_{j,ji} + Bv_{j,ji} + \mu u_{i,jj} + f_i &= b(\dot{u}_j - \dot{v}_j) + \rho_{11}\ddot{u}_i + \rho_{12}\ddot{v}_i, \\ Bu_{j,ji} + Av_{j,ji} + \phi_i &= b(\dot{v}_j - \dot{u}_j) + \rho_{12}\ddot{u}_i + \rho_{22}\ddot{v}_i \end{aligned} \tag{4}$$

s can be related with the interstitial fluid pressure p the single stress variable of interest in fluid,

$$s = \frac{s_{kk}}{3} = -\beta p, \quad s_{ij} = \delta_{ij}s. \tag{5}$$

Under the assumption of one-dimensional problem in the longitudinal direction $\vec{e}_x$ free of external forces per unit volume, stationary in harmonic condition of frequency ω, the solutions of the fluid and the skeleton displacements have the form,

$$u(x,t,\omega) = \tilde{u}e^{i\lambda x}e^{i\omega t}, \quad v(x,t,\omega) = \tilde{v}e^{i\lambda x}e^{i\omega t}, \tag{6}$$

with, $\tilde{u}$ and $\tilde{v}$ constant complex numbers, x the spatial variable in the longitudinal direction, t the time and λ the complex wave number. Expression for the above solutions, deployed in the current work, are given in the literature, e.g. in [2]. Here, we limit ourselves to state that solution is composed by the superimposition of two type of waves the one propagating mainly by the solid skeleton and another one that propagates mainly by the fluid, come to the solution of the form,

$$\nu(x,t,\omega) = \left(C_1\tilde{\nu}^I e^{i\lambda_1 x} + C_2\tilde{\nu}^I e^{-i\lambda_1 x_3} + C_3\tilde{\nu}^{II} e^{i\lambda_3 x} + C_4\tilde{\nu}^{II} e^{-i\lambda_3 x}\right), \tag{7}$$

where ν stands for u or v the solid skeleton and fluid displacements. Finally, to write the explicit expressions of stresses in respect to displacements we introduce the kinematics relations of eq. (1) into the constitutive ones of eq. (2) and considering the case of one–dimensional problems, we get,

$$\sigma = (\lambda' + 2\mu)u_x + Bv_x, \quad s = Bu_x + Av_x. \tag{8}$$

2.1 Developed module

Using well-established analytical solutions from the literature (see [3] referenced in Section 2), we developed an appropriate module in a Java programming environment at Pythmen. `SDE` is an open-source `Java` framework mainly developed by one of the authors (CP). The module can effectively address a configuration of successive sequentially positioning material layers, as shown in Fig. 2.1, interconnected by appropriate interface conditions. The first and the last layer can be considered to extend in the infinite.

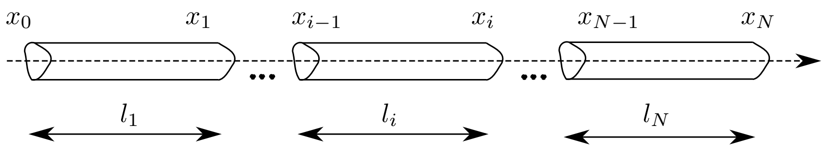

Figure 2.1 *Configuration of successive material layers.*

Each layer of length l_i is supposed to be an one dimensional domain of specific type of fluid, solid or poroelastic medium.

For the fluid domain in the absence of body forces, with compressibility modulus λ_f and density ρ_f, the governing equation is the standard wave equation:

$$\lambda_f w_{xx} = \rho_f \ddot{w}. \tag{9}$$

Furthermore under the assumption of harmonic condition we have the analytic solution of the displacements w, that is given as,

$$w(x,t,\omega) = W_1 e^{i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t} + W_2 e^{-i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t} \tag{10}$$

Boundary conditions referred to a fluid layer are expressed in terms of the displacements w as well as the derivative of displacement in respect to x from which the pressure is derived as,

$$p = -\lambda_f w_x \rightarrow p = -\lambda_f \frac{\partial w}{\partial x}. \tag{11}$$

Explicitly eq. (11) can be written after introducing the,

$$p(x,t,\omega) = -W_1 \left(i\omega\sqrt{\rho_f\lambda_f}\right) e^{i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t} + W_2 \left(i\omega\sqrt{\rho_f\lambda_f}\right) e^{-i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t}. \tag{12}$$

For interconnection conditions between two successive fluid layers we consider the continuity of displacement and equilibrium of pressure, that is,

$$w^i = w^{i+1}, \quad \lambda_f^i \frac{\partial w^i}{\partial x} = -\lambda_f^{i+1} \frac{\partial w^{i+1}}{\partial x}. \tag{13}$$

An elastodynamic solid under the assumption of plane wave in the longitudinal direction $\vec{e}_x$ free of external forces per unit volume, indicating the longitudinal displacement as d, is dictated by,

$$(\lambda_s + 2\mu_s)d_{xx} = \rho_s \ddot{d}, \tag{14}$$

The solution of the above equation under the assumption of harmonic conditions in frequency ω, can be given as,

$$d(x,t,\omega) = D_1 e^{i\omega\sqrt{\frac{\rho_s}{\lambda_s+2\mu_s}}x} e^{i\omega t} + D_2 e^{-i\omega\sqrt{\frac{\rho_s}{\lambda_s+2\mu_s}}x} e^{i\omega t} \tag{15}$$

which is quite similar to eq. (10), while here $lambda_s$ and μ_s the Lame's coefficients for the material of the solid and ρ_s the mass density. Coming to the boundary conditions, when these are present on some solid layer, they expressed in terms of displacements d and the stress component τ which is derived using the displacement's derivatives,

$$\tau = (\lambda_s + 2\mu_s)\frac{\partial d}{\partial x}. \tag{16}$$

Explicitly eq. (16) can be written after introducing the,

$$\tau(x,t,\omega) = -D_1\left(i\omega\sqrt{\rho_f\lambda_f}\right)e^{i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t} + D_2\left(i\omega\sqrt{\rho_f\lambda_f}\right)e^{-i\omega\sqrt{\frac{\rho_f}{\lambda_f}}x} e^{i\omega t}. \tag{17}$$

For the case of two successive solid layers' interconnection we need to consider continuity of displacements and stress equilibrium,

$$d^i = d^{i+1}, \quad (\lambda_s^i + 2\mu_s^i)\frac{\partial d^i}{\partial x} = -(\lambda_s^{i+1} + 2\mu_s^{i+1})\frac{\partial d^{i+1}}{\partial x}. \tag{18}$$

Finally, we mention that poroelastic medium is the most general one, which under certain assumptions can imitate the behaviour of a solid or a fluid therefore capable to degenerate into these types. The analytic solution for this case is the one already given in eq. (7). The boundary conditions in this case are four in number and more specific are expressed in terms of solid skeleton displacements, pore fluid displacements and their derivatives for both of them. Similarly, we should address a two–porous layers interconnection, in the case that we assume that are glued together. In that case the displacement of solid skeleton of the i^{th} poroelastic layer u^i and that of the $i+1$ layer must be equal on that interface. In order to guarantee the equality of interstitial fluid at the interface, the displacement of the fluid at macroscopic level of the two adjacent layers must be inversely proportional to their open porosities:

$$u^i = u^{i+1}, \quad \beta^i v^i = \beta^{i+1} v^{i+1}. \tag{19}$$

Furthermore, we assume that the interstitial pressures of adjacent porous layers are equal and also the same for their total stresses,

$$\beta^{i+1} s^i = \beta^i s^{i+1}, \quad \sigma^i + s^i = \sigma^{i+1} + s^{i+1}. \tag{20}$$

3 Validation

Validation of the current implementation was conducted using analytical solutions for a specific problem configuration that mimics a Kundt's tube experiment. The explicit form of the solution can be found in the literature (e.g., Appendix of [4]). We therefore consider a single poroelastic layer with

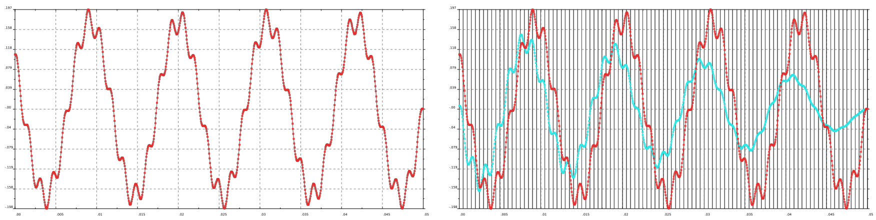

Figure 3.1 *(Left) Solution for solid skeleton's displacement $u(x)$ on the length of the tube x for the test problem. Both literature's analytical solution together with results of the current implementation are plotted.(Right) Computed solution using N=100 individual layers for the the poroelastic medium.*

a rigid, motionless, and impervious end wall, where u=v=0. At the left side, it has the boundary conditions as in eq. (22) for a specific p=$p_0e^{i\omega t}$. The analytical solution for the test poroelastic layer from the literature is shown in Fig. 3.1(Left), along with the results of the current implementation for a single-layer poroelastic medium under excitation at a specific frequency.

In order to validate further the current implementation we keep the same configuration as before yet we split the poroelastic layer to some finite number of individual layers connected to each other through interface conditions presented in here. The actual number of layers is N=100 and the resulting computed response is identical with that of a single layer, depicted in Fig. 3.1(Right). The same figure also shows the solution for the poroelastic medium with gradually increasing Lame's coefficients, demonstrating the potential application of the current implementation.

4 Inverse characterization

Porous fibrous materials are commonly used as acoustic treatments to attenuate noise. To achieve this, it is essential to characterize the properties of such materials. Some of these properties can only be estimated using indirect approaches since it is not possible to measure them directly [5]. Here, we test the feasibility of using the presented one-dimensional analytical solution for the full Biot model, together with the population–based differential evolution global minimization algorithm, to characterize certain material parameters. Given the sound absorption coefficient, which has been numerically produced using the presented analytical solutions, we then try to first estimate the associated to the dynamic coupling of the solid skeleton and the fluid apparent density ρ_{12}

only and then the together all the three of them $\rho_{11}, \rho_{22}, \rho_{12}$. We observed that highly satisfactory estimations can be achieved, and the method demonstrates robustness even in the presence of low-level noise in synthetic measurements.

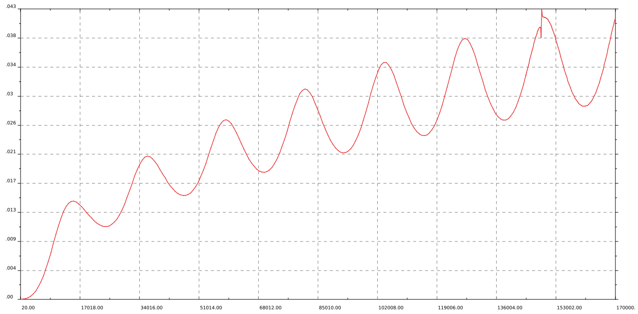

Figure 4.1 *Sound absorption coefficient over a broad range of frequencies, produced using the described analytical solutions for a certain configuration that imitates Kundt's tube.*

5 Conclusions

We have presented the implementation of a module capable of solving problems involving multilayer one-dimensional poroelastic media. For that module analytical solutions existed in the literature have been deployed. Additionally, an inverse acoustical characterization approach for these one–dimensional configurations, using the full Biot theory instead of simplified porous matrix models, has been demonstrated through specific example.

6 References

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